

The dynamo effect - a dynamic renormalisation group approach

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The Dynamo effect is used to describe the generation of magnetic fields in astrophysical objects. However, no rigorous derivation of the dynamo equation is available. We justify the form of the equation using an Operator Product Expansion (OPE) of the relevant fields. We also calculate the coefficients of the OPE series using a dynamic renormalisation group approach and discuss the time evolution of the initial conditions on the initial seed magnetic field.

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The existence of magnetic fields in the astrophysical objects has been an active area of research for quite sometime. This effect is known as the dynamo effect. Many mechanisms have been suggested to explain this. The dynamo effect could be caused by a background turbulent fluid (see, for example [1]) or by a non-turbulent background fluid motion (an interesting example of this type has been discussed in [2]). In this letter we, however, confine ourselves to the study of the first kind.

In turbulent dynamo, to begin with one assumes that initial seed field is very weak i.e. most of the energy is contained in the kinetic part of the magnetised fluid. This immediately tells us that even though the velocity field is influencing the time evolution of the magnetic field, the back reaction of the magnetic field on the velocity field is negligible as the ratio of the Lorentz force to the inertial force is $\sim B^2/u^2$ (this is an order of magnitude estimate of the two nonlinear term in the Navier-Stokes equation) where \mathbf{B} and \mathbf{u} are the magnetic and velocity fields. Basically, the velocity field being unaffected by the magnetic field, will time evolve according to the Navier-Stokes (NS) equation and magnetic field will time evolve according to the Induction equation, being influenced by the velocity field. However, this assumption of weak magnetic field will hold only during the initial transient.

Conventionally, while deriving the dynamo equation one takes a two scale approach [1]: It is assumed that there are two scales of variations -(i) A global scale of variation L of 'mean' quantities and (ii) a scale of small wavelength fluctuations l_o . Mean quantities are defined to be quantities which are averaged over an intermediate scale $a: l_o \ll a \ll L$. In other words, fields have a small wavenumber $\sim L^{-1}$ and a large wavenumber $\sim l_o^{-1}$ components and the mean quantities can be calculated by averaging over the large wavenumbers. We split the velocity and the magnetic fields into mean and fluctuating parts:

$$\mathbf{U}(\mathbf{x}, t) = \mathbf{U}_o(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t), \quad \langle \mathbf{u} \rangle = 0 \quad (1)$$

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_o(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t), \quad \langle \mathbf{b} \rangle = 0 \quad (2)$$

$\langle \rangle$ indicates an averaging over a , the intermediate scale.

The time evolution of the magnetic fields is governed by the Induction equation [1,3]:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \mu \nabla^2 \mathbf{B} \quad (3)$$

where μ is the magnetic viscosity. With the above definitions of the mean and fluctuating quantities, one can easily write down the equations for \mathbf{B}_o and \mathbf{b} . The one for \mathbf{B}_o is given by

$$\frac{\partial \mathbf{B}_o}{\partial t} = \nabla \times (\mathbf{U}_o \times \mathbf{B}_o) + \nabla \times \mathbf{E} + \nu \nabla^2 \mathbf{B}_o \quad (4)$$

where $\mathbf{E} = \langle \mathbf{u} \times \mathbf{b} \rangle$. It is easy to see that, even though $\langle \mathbf{u} \rangle$ and $\langle \mathbf{b} \rangle$ are zero individually, their product may yield a non-zero value because of a possible long range correlation between the two. Then a gradient expansion is performed over this *electromotive force* \mathbf{E} . This has been justified by the argument that \mathbf{E} and \mathbf{B}_o are linearly related:

$$E_i = \alpha_{ij} B_{oj} + \beta_{ijk} \frac{\partial B_{oj}}{\partial x_k} + \gamma_{ijkl} \frac{\partial^2 B_{oj}}{\partial x_k \partial x_l} + \dots \quad (5)$$

In other words, averaging $\mathbf{u} \times \mathbf{b}$ over the intermediate scale gives rise to a gradient expansion of the long wavelength part of the magnetic field. The first term is responsible for ' α -effect' and the second term for ' β -effect'. It can be

easily shown that when the background fluid motion is homogenous and isotropic, α is a pseudo scalar and β is a true scalar. The β term contributes to the effective viscosity ('turbulent diffusion'). Hence, including these α and β effects, the equation for the large scale component of the magnetic fields becomes

$$\frac{\partial \mathbf{B}_o}{\partial t} = \nabla \times (\mathbf{U}_o \times \mathbf{B}_o) + \alpha \nabla \times \mathbf{B}_o + \eta \nabla^2 \mathbf{B}_o \quad (6)$$

where $\eta = \nu + \beta$, the effective viscosity. This is the dynamo equation.

Even though this equation has been very successful, the justification for the existence of a gradient expansion (which is a crucial part in the derivation) is not founded upon a strong basis. Here we attempt to give a rigorous basis to it: From a field theoretic point of view $\langle \mathbf{u}(\mathbf{r}_1) \times \mathbf{b}(\mathbf{r}_2) \rangle$ (here $\mathbf{u}(\mathbf{r}_1)$ and $\mathbf{b}(\mathbf{r}_2)$ are the 'fluctuating' part of the total velocity and the magnetic field as defined in equations (1) and (2)) diverges as $r_1 \rightarrow r_2$. Hence, one can write down an Operator Product Expansion (OPE) [4] for the product of the fields $Lt_{r_1 \rightarrow r_2} u(r_1) \times b(r_2)$ and consequently calculate $Lt_{r_1 \rightarrow r_2} \langle \mathbf{u} \times \mathbf{b} \rangle$. Since the LHS is linear in \mathbf{b} , the RHS will have only odd powers of \mathbf{B}_o (so that $\mathbf{B} \rightarrow -\mathbf{B}$ property is retained). For the same reason, there cannot be any term having no \mathbf{B}_o . Also, no term proportional to \mathbf{b} can appear as $\mathbf{b} \cdot \mathbf{u} \times \mathbf{b} = \mathbf{0}$ (the LHS). However, no such condition is there for the velocity field as if \mathbf{u} is a solution of the NS equation then $-\mathbf{u}$ is not. We write down the OPE as

$$(\mathbf{u} \times \mathbf{b})_i \equiv E_i = \alpha_{ij} B_{oj} + \beta_{ijk} \frac{\partial B_{oj}}{\partial x_k} + \dots + \square \quad (7)$$

where \square indicates various composite operators permitted by the symmetry of the LHS. Since, expectation value of any composite operator is defined to be zero we have

$$\langle \mathbf{u} \times \mathbf{b} \rangle = \alpha_{ij} B_{oj} + \beta_{ijk} \frac{\partial B_{oj}}{\partial x_k} + \dots \quad (8)$$

which is in agreement with the form of the equation.

In k -space, $Lt_{r_1 \rightarrow r_2} \langle \mathbf{u} \times \mathbf{b} \rangle = \int_q \mathbf{u}(\mathbf{q}) \times \mathbf{b}(\mathbf{k} - \mathbf{q})$. Here, we take k to be small i.e. long wavelength limit and $\Lambda > q > \Lambda e^{-r}$ with r positive, i.e. q belongs to some short wavelength band. The RHS becomes, $\int_q \alpha_{ij}(q) B_{oj}(k - q) + \int_q \beta_{ijk}(q) k_j B_{ok}(k - q) + \dots$ (in principle, α, β etc. could be functions of k). For calculating the OPE coefficients, we calculate the LHS by momentum shell integration, do a loop expansion and equate various powers of k on both sides. We work in the incompressible limit.

The Navier-Stokes equation is given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \nu \nabla^2 \mathbf{u} - \frac{\nabla p}{\rho} + \mathbf{f} \quad (9)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (10)$$

The last one is the incompressibility assumption. Here, ν is the fluid viscosity, p the pressure and ρ the density. This in k -space reduces to

$$\frac{\partial u_\alpha(k, t)}{\partial t} + \frac{i}{2} P_{\alpha\beta\gamma}(k) \int_q u_{\beta\gamma}(q) u_k(k - q) = -\nu k^2 u_\alpha + f_\alpha \quad (11)$$

where $P_{\alpha\beta\gamma} = P_{\alpha\beta} k_\gamma + P_{\alpha\gamma} k_\beta$ and $P_{\alpha\beta}$ is the projection operator. Since we are looking at turbulent dynamo, we take $\langle f_\alpha(k, t) f_\beta(k', t') \rangle = D_1(k) P_{\alpha\beta} \delta(k + k') \delta(t - t') + D_2(k) \epsilon_{\alpha\beta\gamma} k_\gamma \delta(k + k') \delta(t - t')$. D_1 and D_2 are even in k . An explicit factor of i in front of D_2 term indicates that it is an odd parity breaking term and according to our previous analysis α -effect will be absent if $D_2 = 0$.

The induction equation is given by

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{b}) + \mu \nabla^2 \mathbf{b} \quad (12)$$

This, in k -space becomes

$$\frac{\partial b_\alpha(k, t)}{\partial t} = i \epsilon_{\alpha\beta\gamma} k_\beta \epsilon_{\gamma\mu\lambda} \int_q u_\mu(q) b_\lambda(k - q) - \mu k^2 b_\alpha \quad (13)$$

At the tree level (fig.1), the diagram which will contribute is the following: We have, $\langle (\mathbf{u} \times \mathbf{B})_\alpha \rangle = \langle$

$i \int_{q,\Omega} \epsilon_{\alpha\beta\gamma} \epsilon_{\gamma\delta\lambda} \epsilon_{\lambda\eta\tau} (k-q)_\delta [D_1(q) P_{\beta\eta}(q) + D_2(q) \epsilon_{\beta\eta\rho} q_\rho] G_o^b(k-q, \omega-\Omega) C_o^u(q, \omega) (k-q)_\delta B_\tau(k)$, where $G_o^b(q, \Omega) = \frac{1}{i\Omega + \mu q^2}$ and $C_o^u(q, \Omega) = \frac{1}{\Omega^2 + \nu^2 q^4}$. We extract the α -term and the β term from the above expression in the following way:

I. The α term: In the above integral, there is a term which is independent of k . This gives rise to the α term

$$\alpha B_\alpha(k) = \int d^3q d\Omega \epsilon_{\alpha\beta\gamma} \epsilon_{\gamma\delta\lambda} \epsilon_{\lambda\eta\tau} \epsilon_{\beta\eta\rho} \frac{(k-q)_\delta q_\rho D_2(q)}{\Omega^2 + \nu^2 q^4} \frac{1}{i(\omega - \Omega) + \mu(k-q)^2} B_\tau(k) \quad (14)$$

Now, $\epsilon_{\alpha\beta\gamma} \epsilon_{\gamma\delta\lambda} \epsilon_{\lambda\eta\tau} \epsilon_{\beta\eta\rho} = \delta_{\alpha\delta} \delta_{\tau\rho} + \delta_{\alpha\rho} \delta_{\tau\delta}$. Hence, one obtains

$$\alpha = \frac{4\pi^2}{3} \frac{1}{\nu(\nu + \mu)} \int_q D_2(q) \quad (15)$$

Since, the integral is infra-red divergent, we evaluate it by shell integration.

II. The β term: The term proportional to k gives rise to the β term

$$i\beta \epsilon_{\lambda\delta\tau} = i \int d^3q d\Omega D_1(q) \epsilon_{\alpha\beta\gamma} \epsilon_{\gamma\delta\lambda} \epsilon_{\lambda\eta\tau} P_{\beta\eta}(q) (k-q)_\delta \quad (16)$$

Now, $\epsilon_{\alpha\beta\gamma} \epsilon_{\gamma\delta\lambda} = \delta_{\alpha\delta} \delta_{\beta\lambda} - \delta_{\alpha\lambda} \delta_{\beta\delta}$. Hence,

$$\beta = \frac{4\pi^2}{3} \frac{1}{\nu(\nu + \mu)} \int_q \frac{D_1(q)}{q^2} \quad (17)$$

Notice that α is pseudo-scalar and β is a true scalar. We see that even for thermal noise i.e D_1 and D_2 being constants give nonvanishing α and β . However, if we assume that the background fluid is fully developed turbulent, then $D_1(k) \sim k^{-3}$ and $D_2(k) \sim k^{-4}$ (this particular choice gives $E(k) \sim k^{-5/3}$ [5]).

Having calculated α and β from the tree level diagram, we now analyse the higher-loop diagrams. There are several distinct classes of them: (i) This type can be generated by decorating the velocity correlator or the response function. They generate corrections to the coefficients already obtained without decorations (fig 2a). (ii) This type of diagrams are generated by decorating the magnetic field response function. They give rise to non-zero values for the higher order coefficients of the OPE series as well as corrections to the coefficients already obtained without decorations (fig 2b). These two classes of diagrams can be obtained by considering a 'self-consistent' tree level diagram.

(iii) These type of diagrams appear by inserting two ub vertex (the advective vertex of the Induction equation) in the tree level diagram (fig 3a) or by inserting one uu and one ub vertex (fig 3b and 3c). A 1-loop diagram as shown in fig 3a is given as $\int_{q,p} G_o^b(k-q) C_o^u(q) A_1(p)$ where

$$A_1(p) = G_o^b(k-q-p) G_o^b(k-p) C_o^u(p) \quad (18)$$

Similarly, fig 3b and fig 3c are given by $\int_{q,p} G_o^b(k-q) C_o^u(q) A_2(p)$ and $\int_{q,p} G_o^b(k-q) C_o^u(q) A_3(q)$ where

$$A_2(p) = G_o^b(k-p) G_o^u(p-q) C_o^u(p) \quad (19)$$

$$A_3(p) = G_o^b(k-p) G_o^u(-p) C_o^u(p-q) \quad (20)$$

We see that the sum of A_1, A_2 and A_3 are nothing but the 1-loop correction of the advective vertex of the Induction equation. Sum of A_1, A_2 and A_3 are however not zero as we are calculating the vertex correction at finite external momenta. These diagrams give rise to 1-loop corrections to α and β (which are already present in the tree level) as well as the lowest order γ . It is easy to see that an n -loop diagram will give rise to terms upto the n -th term in the OPE series starting from α .

A pertinent question at this stage is: How does initial correlations among the seed magnetic field (a dynamo is essentially an initial value problem) grow? Also, if the initial magnetic field correlations are parity symmetric, will it generate a parity breaking correlation after a finite time (possibly due to the parity breaking correlation of the forcing in the Navier-Stokes equation)? Consider a 1-loop diagram which would renormalise the initial magnetic field correlations: It is clear that inside such a 1-loop integral, one noise correlation arises from the NS equation and the second one is from the initial magnetic field correlation. Hence, if the N-S noise has a nonvanishing parity breaking term, it will give a non-zero contribution to the 1-loop integral only if the initial seed magnetic field correlation also has a parity breaking term, as one requires two or even number 'epsilon' terms for the integral not to vanish. Consequently, we can say that unless the initial correlation has a parity breaking term, it will not be generated in the

future time. It is also clear that even if the initial magnetic field correlation is non-singular at $t = 0$, it will pick up a singular contribution at a later time (since the noise in the NS equation has a singular correlation).

It is clear that due to the dynamo action, the initial magnetic field will grow. Now, once $\int B^2 / \int v^2$ becomes ~ 1 , it is no longer justified to neglect the Lorentz force term, which we have neglected so far. Adding this term to the N-S equation we have the following set of equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \nu \nabla^2 \mathbf{u} + \frac{1}{4\pi\rho} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{\nabla p}{\rho} + \mathbf{f} \quad (21)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (22)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \alpha \nabla \times \mathbf{B} + \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \quad (23)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (24)$$

We choose

$$\langle f_i(k, t) f_j(k', t') \rangle = [D_1 P_{ij} k^{-3} + i D_2 \epsilon_{ijp} k_p] \delta(k + k') \delta(t - t') \quad (25)$$

$$\langle B_i(k, t=0) B_j(k', t=0) \rangle = D P_{ij} k^{-3} \delta(k + k') \quad (26)$$

It is clear the dynamo equation has two linear terms - one proportional to k and the other proportional to k^2 . Conventionally, one may like to keep all the linear term in the bare response function. However, that would give rise to a singularity at a finite real k . To avoid that problem, we keep only the viscosity term in the bare response function and treat the α term perturbatively. Following Forster *et al* [6] we obtain the following recursion relations for the parameters:

$$\frac{d\nu}{dr} = \nu(r) [z - 2 + A_1 \frac{D_1}{\nu^3 \Lambda^4} + A_2 \frac{D_2}{\nu \eta^2 \Lambda^4}] \quad (27)$$

$$\frac{d\eta}{dr} = \eta(r) [z - 2 + A_3 \frac{D_1}{\nu(\nu + \eta) \Lambda^4} + A_4 \frac{D_2}{\eta(\eta + \nu) \Lambda^4} + A_5 \frac{\alpha^2}{\eta \Lambda^2}] \quad (28)$$

$$\frac{d\alpha}{dr} = [z - 1 + A_6 \frac{1}{\alpha \nu (\nu + \eta) \Lambda^{1/3}}] \quad (29)$$

One can see easily that $\alpha \sim k^{-1/3}$ and $\eta \sim k^{-4/3}$. A_1, \dots, A_5 are numerical constants which determine the non-universal amplitudes. It is clear that all the exponents dynamic ($=2/3$) and roughness exponents are same as usual MHD turbulence. Only the nonuniversal amplitudes are different. Here η is the effective viscosity (or the 'turbulent diffusion' as opposed to the bare molecular viscosity. This is the origin of the enhanced diffusion process in many astrophysical situations e.g accretion disk [7].

So, in conclusion, we have shown how the dynamo equation can be formally constructed using OPE. We have shown how to calculate the coefficients of the OPE series perturbatively in a loop expansion. In particular we have evaluated α and β in a momentum shell elimination method. We have also discussed renormalisation of the dynamo equation.

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Figure Captions: Fig 1: Tree level diagrams for $\langle \mathbf{u}(\mathbf{q}) \times \mathbf{b}(\mathbf{k} - \mathbf{q}) \rangle$. A solid line indicates a bare magnetic field response function, a broken line indicates a bare velocity response function, a 'o' joined by two broken lines indicates a bare velocity correlation function, a wavy line indicates a magnetic field, a solid triangle indicates a ub vertex.

Fig 2a and 2b: 1-loop corrections to the previous tree level diagram which arise due to dressing of the bare velocity response/correlation functions and bare magnetic field response function, a 'X' indicates a uu vertex. Other symbols have same meaning.

Fig 3a, 3b and 3c: 1-loop correction to tree level diagram arising out of 1-loop corrections to the ub vertex. Symbols have same meaning.

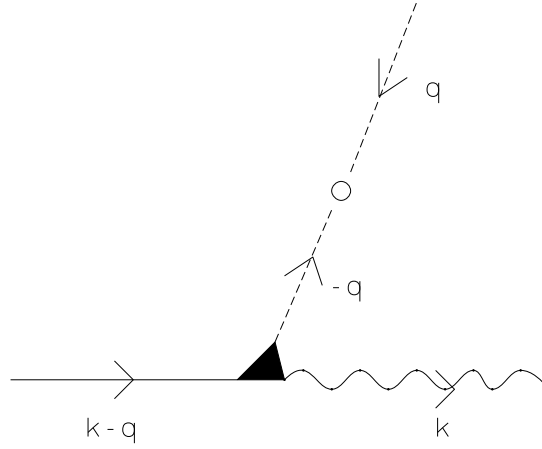


FIG. 1.

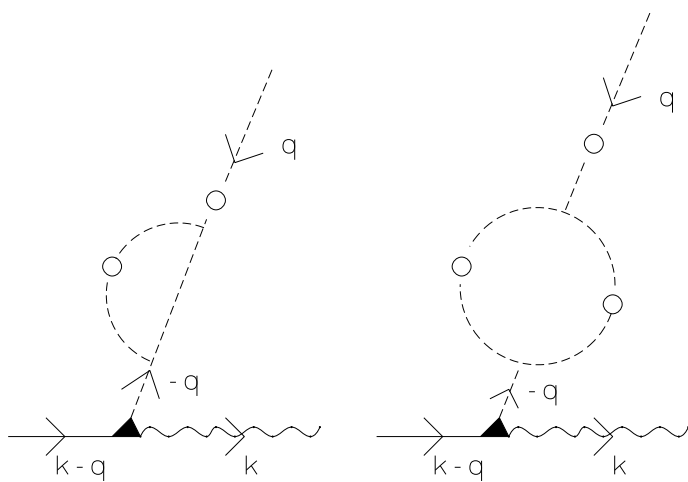


Figure - 2a

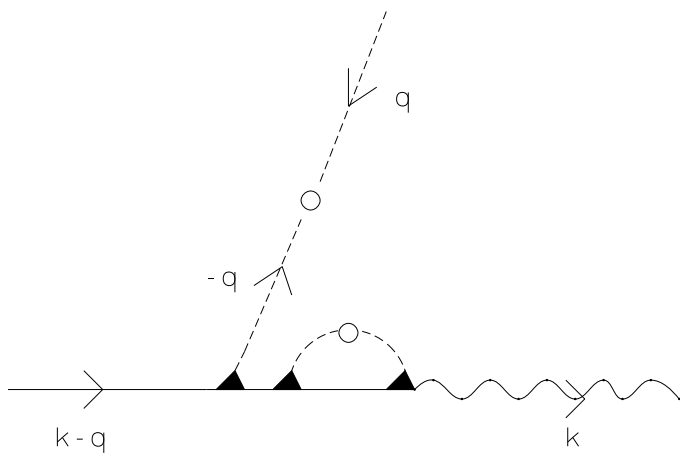


Figure - 2b

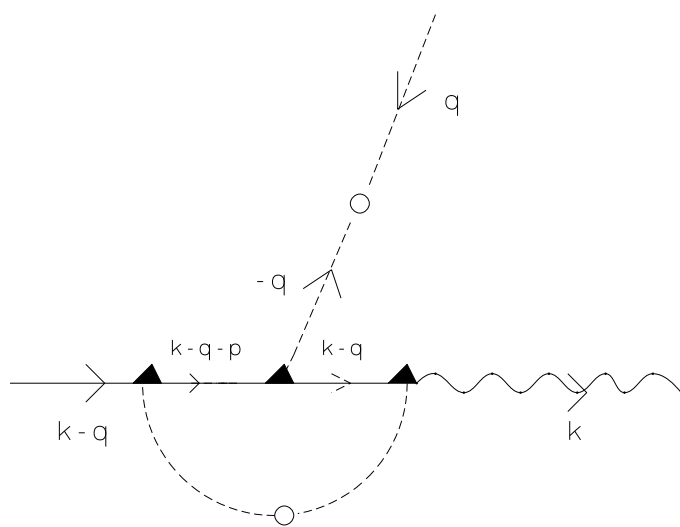


Figure - 3a

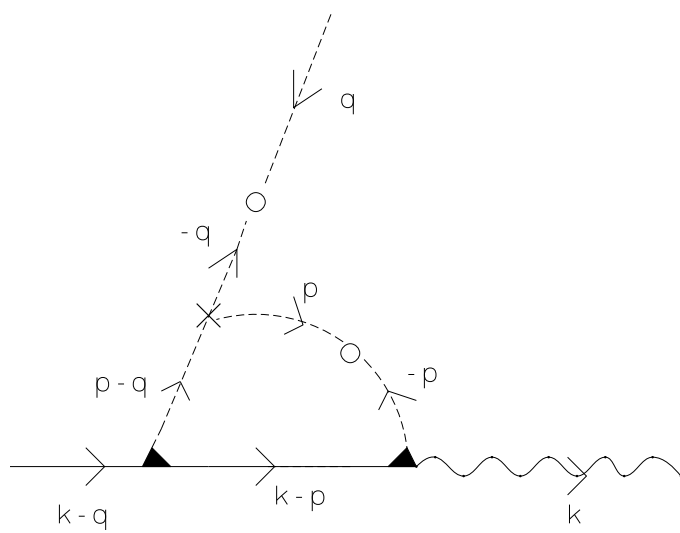


Figure - 3b

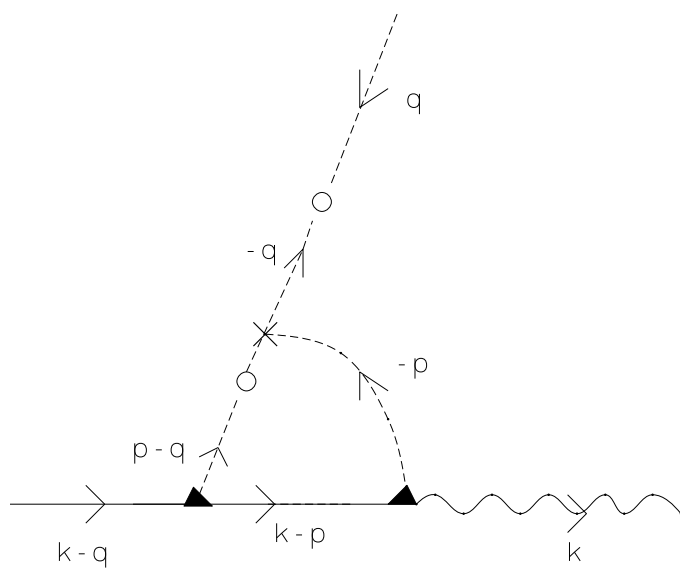


Figure - 3c